Boolean Algebras and Category Theory

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Abstract

These notes to a talk define Boolean Algebras in the language of Category theory. In order to show the Tarski-Lindenbaum-Duality, completeness and atoms are discussed apart from basic properties of Boolean Algebras. There are also some (implicit) examples of how Category theory gives some properties "for free".

1 Preorders

Recall that a *preorder* is a category with at most one morphism between two objects. A *partial order* is a preorder where isomorphic objects are equal.

In a preorder, we write $a \leq b$ if there is a morphism from object a to object b.

Functors between preorders are exactly monotone functions; covariant if increasing, contravariant if decreasing.

2 Boolean Algebras

Definition. A partial order $(\mathcal{B}, \leq)$ is called a *Boolean Algebra*, if the following conditions hold:

1. It has an initial and a terminal object, denoted 0 and 1.
2. It has binary products and coproducts, denoted $\wedge$ and $\vee$.
3. For all $x \in \mathcal{B}$, the functor $x \wedge \cdot$ has a right adjoint and the functor $x \vee \cdot$ has a left adjoint.
4. There is a *injective* functor $\neg : \mathcal{B} \to \mathcal{B}^{\circ}$, mapping each object to its *complement*, which has a right adjoint and for which $\neg a \leq a$ implies $a = 1$.

Remark. Notice that the functors and adjoints we ask for are functors of preorders, but the functors of Boolean Algebras are also exactly monotone functions.

This definition uses categorical notions, but it is equivalent to the "classical" (non-categorical) notion of Boolean Algebra. Part 1 and 2 simply say there is a least and a greatest element and that any two elements have a greatest common lower bound and a least common upper bound. These have the following (obvious) properties for any a in $\mathcal{B}$:

$$a \wedge a = a, \quad a \vee a = a, \quad a \wedge 1 = a, \quad a \vee 0 = a$$

Proposition 2.1. *Boolean Algebras are distributive with respect to both $\wedge$ and $\vee$.*

Proof. $x \wedge \cdot$ and $x \vee \cdot$ are in fact functors of preorders. It is sufficient to show they are order-preserving:

$$a \leq b \Rightarrow x \wedge a \leq b\ , \quad x \wedge a \leq x \implies x \wedge a \leq x \wedge b$$

$$a \leq b \Rightarrow a \leq x \vee b\ , \quad x \leq x \vee b \implies x \vee a \leq x \vee b$$

By condition 3, $x \wedge \cdot$ is a left adjoint, thus it preserves coproducts and $x \vee \cdot$ is a right adjoint, thus it preserves products. This gives exactly distributivity of $\wedge$ and $\vee$. □

Proposition 2.2. *For every object x in $\mathcal{B}$, we have $\neg x \wedge x = 0$ and $\neg x \vee x = 1$.*

Proof. We ask $\neg$ to be a functor, that is an order-reversing function. So we obtain

$$a \leq b \implies \neg b \leq \neg a.$$

For b in $\mathcal{B}$ let a such that $b \leq a$ and $\neg b \leq a$. The above gives $\neg a \leq \neg b$, with $\neg b \leq a$ follows $\neg a \leq a$ and $a = 1$. We obtain $\neg b \vee b = 1$.

As $\neg$ is a left adjoint, it preserves colimits, in particular the initial objects. As colimits become limits in the dual category, it follows immediately that $\neg 0 = 1$.

Let now $b \leq \neg b$. As $\neg$ is order-reversing, we have $\neg(\neg b) \leq \neg b$. Putting $a = \neg b$, $\neg a \leq a$ implies that $\neg b = a = 1 = \neg 0$. E.g. by injectivity, we obtain $b = 0$.

Now consider again any b in $\mathcal{B}$. Let a such that $a \leq b$ and $a \leq \neg b$. The order-reversing property gives $\neg b \leq \neg a$, thus $a \leq \neg a$ which implies by the above $a = 0$ and we have $\neg a \wedge a = 0$. □

Proposition 2.3. *The complement $\neg a$ is the unique b which satisfies $b \wedge a = 0$ and $b \vee a = 1$.*

Proof. Let b as in the proposition. Now by using distributivity:

$$b = b \wedge 1 = b \wedge (\neg a \vee a) = (b \vee \neg a) \wedge (b \vee a) = b \vee \neg a \geq \neg a$$

We obtain in the same way that $\neg a \geq b$. So $\neg a$ and b are isomorphic which implies $\neg a = b$. □

Proposition 2.4. $\neg\neg = \mathrm{id}_{\mathcal{B}}$, *thus* $\neg$ *is a bijection and self-adjoint. It follows that* $\neg$ *preserves coproducts* and *products.*

Proof. The preceding proposition implies immediately that $\neg(\neg a) = a$ for all a in $\mathcal{B}$ by the symmetry of the conditions for the complement. So $\neg\neg \equiv \mathrm{id}_{\mathcal{B}}$ and $\neg$ is in fact self-adjoint. Order-reversing gives $\neg a \leq b \Leftrightarrow \neg b \leq \neg\neg a = a$ and thus a natural isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(\neg a, b) \simeq \mathrm{Hom}_{\mathcal{B}^\circ}(a, \neg b)$$

Naturality is evident as the Hom-Set contains at most one element. So $\neg$ is both a left *and* right adjoint and thus preserves limits and colimits. This gives immediately De Morgan's laws:

$$\neg(a \wedge b) = \neg a \vee \neg b \quad \text{and} \quad \neg(a \vee b) = \neg a \wedge \neg b.$$

□

Equivalence with standard definition

Common definitions for Boolean Algebras require a partial order with a smallest and greatest element and for any two elements, a least common upper bound and a greatest common lower bound. These correspond with part 1 and 2 of our definition. Additionally, the partial order has to be distributive and provide a complement $\neg x$ for every element x satisfying $\neg x \wedge x = 0$ and $\neg x \vee x = 1$. As shown above, one obtains $\neg\neg x = x$ by unicity of the complement.

We have already shown that our definition gives a structure which satisfies these classical conditions. But we can also show that any structure satisfying the classical conditions is a Boolean Algebra as defined above.

$$x \wedge a \leq b \Rightarrow \neg x \vee (x \wedge a) \leq \neg x \vee b \Rightarrow (\neg x \vee x) \wedge (\neg x \vee a) \leq \neg x \vee b \Rightarrow a \leq \neg x \vee a \leq \neg x \vee b$$

$$a \leq \neg x \vee b \Rightarrow x \wedge a \leq x \wedge (\neg x \vee b) \Rightarrow x \wedge a \leq (x \wedge \neg x) \vee (x \wedge b) \Rightarrow x \wedge a \leq x \wedge b \leq b$$

This gives adjunctions for $x \wedge \cdot$ and $x \vee \cdot$ (remember these are order-preserving):

$$\mathrm{Hom}_{\mathcal{B}}(x \wedge a, b) \simeq \mathrm{Hom}_{\mathcal{B}}(a, \neg x \vee b) \quad (1)$$
$$\mathrm{Hom}_{\mathcal{B}}(\neg x \wedge a, b) \simeq \mathrm{Hom}_{\mathcal{B}}(a, x \vee b) \quad (2)$$

Now show that $\neg$ is order-reversing:

$$a \leq b \Rightarrow \neg b \wedge a \leq \neg b \wedge b = 0 \Rightarrow \neg a = \neg a \vee (\neg b \wedge a) = (\neg a \vee \neg b) \wedge (\neg a \vee a) = \neg a \vee \neg b \Rightarrow \neg b \leq \neg a$$

This gives an adjunction by the self-inverse property:

$$\mathrm{Hom}_{\mathcal{B}}(\neg a, b) \simeq \mathrm{Hom}_{\mathcal{B}^\circ}(a, \neg b) \quad (3)$$

Finally, we check that $\neg a \leq a$ implies $a = 1$ by $a \geq a \vee \neg a = 1$.

3 Completeness and Atoms

Definition. A Boolean Algebra is called *complete*, if it is complete and cocomplete as a category.

Remark. In fact, all that is needed are small products and coproducts, but the existence of those makes the category (co)complete by a proposition from the lecture.

For any $A \subset \mathcal{B}$, we write $\bigwedge A$ for the product and $\bigvee A$ for the coproduct of all elements of A. Additionally, we allow the notations

$$\bigwedge A = \bigwedge_{a \in A} a \quad \text{and} \quad \bigvee A = \bigvee_{a \in A} a.$$

We also notice these relations for $A, B \subset \mathcal{B}$:

$$\bigwedge A \ \wedge \ \bigwedge B = \bigwedge (A \cup B) \quad \text{and} \quad \bigvee A \ \vee \ \bigvee B = \bigvee (A \cup B)$$

$$\text{the above implies:} \quad A \subset B \implies \bigwedge A \leq \bigwedge B \text{ and } \bigvee A \leq \bigvee B$$

Definition. An object a in a Boolean Algebra $\mathcal{B}$ is called an *atom*, if $a \neq 0$ and $b \leq a$ implies $b = 0$ or $b = a$.

Definition. For $A \subset \mathcal{B}$, we put $\text{Atoms}(A) = \{a \,\text{atom} | a \leq \bigvee A\}$, that is the set of atoms below the coproduct of A.

Remark. If we put $A = \mathcal{B}$, certainly is $1 \in A$. Thus $\bigvee A = 1$ and thus $\text{Atoms}(A)$ is exactly the set of all atoms in $\mathcal{B}$.

Definition. A Boolean Algebra $\mathcal{B}$ is called *atomic*, if for every $x \in \mathcal{B}$ we have $x = \bigvee \text{Atoms}(x)$.

Remark. This allows us to think of $\text{Atoms}(\mathcal{B})$ as the set of generators of $\mathcal{B}$ – but we will put this more precisely later.

Obviously $1 = \bigvee \text{Atoms}(\mathcal{B})$ holds in an atomic Boolean Algebra.

Proposition 3.1. *Let $\mathcal{B}$ an atomic Boolean Algebra. If $c \in \text{Atoms}(\mathcal{B})$, $A \subset \mathcal{B}$ and $c \leq \bigvee A$ then there exists an $a \in A$ such that $c \leq a$ (i.e. we can pull atoms back).*

Proof. We obtain this from the adjunction of $x \wedge \cdot$

$$a \wedge b = 0 \Leftrightarrow b \leq \neg a.$$

Now for c an atom, let $c \not\leq a$. It follows $c \wedge a = 0$ from $c \wedge a \leq c$. On the other hand, if $c \wedge a = 0$, then as $c \leq a$ implies $c = c \wedge a$, we have $c \not\leq a$. This yields

$$c \not\leq a \Leftrightarrow a \leq \neg c.$$

In order to prove the proposition, assume that $c \not\leq a$ for all $a \in A$. Then $a \leq \neg c$ for all $a \in A$, thus $\bigvee A \leq \neg c$. But his means $c \not\leq \bigvee A$, which contradicts the assumption $c \leq \bigvee A$. □

4 Homomorphisms of Boolean Algebras

Definition. A map $F : \mathcal{B} \to \mathcal{C}$ is called a *homomorphism* of Boolean Algebras $\mathcal{B}$ and $\mathcal{C}$ if it preserves small products and coproducts (including empty ones).

Remark. Homomorphisms preserve 0 and 1,the empty coproduct and product, and complements, as those are characterised by $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

Homomorphisms are also order-preserving, as $a \wedge b = b \Leftrightarrow b \leq a$. So homomorphisms are functors (But functors are not necessarily homomorphisms[1]).

One very prominent example of a homomorphism is $\neg$ by its adjoint properties. Moreover, as it is self-inverse, it is actually an isomorphism, thus $\mathcal{B} \simeq \mathcal{B}^\circ$. This tells us that $\text{Atoms}(\mathcal{B}^\circ) = \neg\text{Atoms}(\mathcal{B})$.

Proposition 4.1. *If a map $f : \mathcal{B} \to \mathcal{C}$ between Boolean algebras preserves coproducts and complements, it is a homomorphism.*

Proof. Let $A \subset \mathcal{B}$. We write $f(A) = \{f(a) | a \in A\}$ for any f and by the adjoint and self-inverse properties of $\neg$ follows:

$$f\left(\bigwedge A\right) = f\left(\neg \bigvee \neg A\right) = \neg \bigvee \neg f(A) = \bigwedge f(A)$$

□

Proposition 4.2. CABool *with Complete Atomic Boolean Algebras as objects and homomorphisms between them as arrows is a category.*

Proof. The existence of an identity for every CABA is pretty obvious and so are the laws of associativity and identity if the composition of homomorphisms is well defined by composition of maps.

So let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ two homomorphisms. Show that $G \circ F$ preserves coproducts and products:

Let $A \subset \mathcal{A}$.

$$G \circ F\left(\bigvee A\right) = G\left(\bigvee F(A)\right) = \bigvee G \circ F(A)$$

Analogically for the product. □

5 The Tarski-Lindenbaum-Duality

Theorem 1. CABool *is equivalent to* Sets$^\circ$.

Remark. In this section Boolean Algebras are always complete and atomic.

Proposition 5.1. *For a set X, the powerset $\mathbb{P}(X)$ ordered with inclusion is a Complete Atomic Boolean Algebra, where the atoms are exactly the singleton subsets of X.*

A map of sets $f : Y \to X$ gives a homomorphism of Boolean Algebras by the inverse image $f^{-1} : \mathbb{P}(X) \to \mathbb{P}(Y)$.

[1]e.g. consider the map from any $\mathcal{B}$ to $\{0, 1\}$ which sends everything to 1.

Proof. The first statement is self-evident by the definition of Set Algebra,[2] where the intersection is the product, the union is the product and the complement is – of course – the set complement in X. The empty set is the initial, X itself is the terminal object.

For the second part, expand the definition of $f^{-1} : \mathbb{P}(X) \to \mathbb{P}(Y)$:

$$A \subset X \qquad f^{-1}(A) = \{b \in Y | \exists a \in A \,\text{with}\, f(b) = a\}$$

Now consider a family $(A_i)_{i \in I}$ of subsets of X and check that union and intersection are stable under f^{-1}.

$$\begin{aligned} f^{-1}\left(\bigcup_{i\in I} A_i\right) = \left\{b \in Y \,\middle|\, \exists a \in \bigcup_{i\in I} A_i : f(b) = a\right\} &= \{b \in Y \,| \exists i \in I : \exists a \in A_i : f(b) = a\} \\ &= \bigcup_{i\in I} \{b \in Y \,| \exists a \in A_i : f(b) = a\} = \bigcup_{i\in I} f^{-1}(A_i) \end{aligned}$$

$$\begin{aligned} f^{-1}\left(\bigcap_{i\in I} A_i\right) = \left\{b \in Y \,\middle|\, \exists a \in \bigcap_{i\in I} A_i : f(b) = a\right\} &= \{b \in Y \,| \forall i \in I : \exists a \in A_i : f(b) = a\} \\ &= \bigcap_{i\in I} \{b \in Y \,| \exists a \in A_i : f(b) = a\} = \bigcap_{i\in I} f^{-1}(A_i) \end{aligned}$$

□

This gives a *contravariant* functor $F : \text{Sets} \to \text{CABool}$ which assigns to every set X the Boolean Algebra $\mathbb{P}(X)$ and to every map $f : Y \to X$ between sets the homomorphism from the above Proposition $f^{-1} : \mathbb{P}(X) \to \mathbb{P}(Y)$. So F is a *covariant* functor from Sets° to CABool.

Proposition 5.2. *There is a functor $G : \text{CABool} \to \text{Sets}^\circ$ which assigns to every Boolean Algebra its set of atoms.*

Proof. Obviously we have to define what G should do with homomorphisms such that it becomes a functor. So let $g : \mathcal{B} \to \mathcal{C}$ a homomorphism of Boolean Algebras. We set $G(g) : G(\mathcal{C}) \to G(\mathcal{B})$ by putting $G(g)(c) = b$ for all $c \in G(\mathcal{C})$ and where b is the unique $b \in G(\mathcal{B})$ with $c \leq g(b)$.

This b exists, because $1_\mathcal{B} = \bigvee \text{Atoms}(\mathcal{B})$ and consequently $c \leq 1_\mathcal{C} = g(1_\mathcal{B}) = \bigvee\{g(b) | b \in \text{Atoms}(\mathcal{B})\}$ gives by proposition 3.1, that there is $g(b)$ with $c \leq g(b)$. It is also unique, because if $b_1 \neq b_2$ are atoms in $\mathcal{B}$ and $c \leq g(b_1)$ and $c \leq g(b_2)$, then

$$c \leq g(b_1) \wedge g(b_2) = g(b_1 \wedge b_2) = g(0) = 0.$$

[2]Set Algebra is more or less defined in the same way as Boolean Algebra.

If $g = \mathrm{id}_{\mathcal{B}}$, it follows that $G(g)(c) = c$ for all $c \in \mathrm{Atoms}(\mathcal{B})$, so $G(\mathrm{id}_{\mathcal{B}}) = \mathrm{id}_{\mathrm{Atoms}(\mathcal{B})}$. For $g : \mathcal{A} \to \mathcal{B}$ and $g' : \mathcal{B} \to \mathcal{C}$ homomorphisms we have

$$\begin{aligned} &G(g)(b) = a \text{ with } b \leq g(a), \quad G(g')(c) = b \text{ with } c \leq g'(b) \\ &\qquad \text{so } G(g) \circ G(g')(c) = a \text{ with } c \leq g'(g(a)) \\ &\qquad\qquad \text{as in } G(g' \circ g)(\tilde{c}) = \tilde{a} \text{ with } \tilde{c} \leq (g' \circ g)(\tilde{a}) \end{aligned}$$

So G is indeed a functor. □

Proof. (of Theorem 1)

In order to show that the functors F and G give an equivalence (or rather duality) between categories, we have to show that we have natural isomorphisms between the composites $F \circ G$, $G \circ F$ and the respective identity functors.

For any X in Sets° we have

$$X \xrightarrow{F} \mathbb{P}(X) \xrightarrow{G} \mathrm{Atoms}(\mathbb{P}(X)) = \{\{x\} | x \in X\}$$

Clearly, the set of singleton subsets is isomorphic to the set itself. Obviously, this gives a natural isomorphism $\mathrm{id}_{\mathrm{Sets}^\circ} \xrightarrow{\sim} G \circ F$.

Furthermore, we need a natural isomorphism $\eta : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathrm{CABool}}$:

$$\begin{array}{rccc} \eta_{\mathcal{B}} : & \mathbb{P}(\mathrm{Atoms}(\mathcal{B})) & \to & \mathcal{B} \\ & A & \mapsto & \bigvee A \end{array}$$

This map is certainly bijective by definition of a *complete atomic* Boolean Algebra which says that every object in $\mathcal{B}$ is exactly the coproduct of the atoms below it and that all coproducts (of atoms) exist.

Consider a family $(A_i)_{i \in I}$ of subsets of $\mathrm{Atoms}(\mathcal{B})$. Keep in mind the a's are all atoms:

$$\exists i \in I,\ a \in A_i \Leftrightarrow a \leq \bigvee_{i \in I} \bigvee A_i \quad \Longrightarrow \quad \bigvee_{i \in I} \bigvee A_i = \bigvee \bigcup_{i \in I} A_i$$

$$\forall i \in I,\ a \in A_i \Leftrightarrow a \leq \bigwedge_{i \in I} \bigvee A_i \quad \Longrightarrow \quad \bigwedge_{i \in I} \bigvee A_i = \bigvee \bigcap_{i \in I} A_i$$

Thus $\eta_{\mathcal{B}}$ is a homomorphism of Boolean Algebras.

Finally, we have to check that η is natural. But any homomorphism between atomic BA's is fully determined by the images of the atoms, so such a homomorphism gives immediately a homomorphism on $\mathbb{P}(\mathrm{Atoms}(\mathcal{B}))$. □

Remark. This nice Theorem 1 says that Complete Boolean Algebras are exactly what we think they are – just powersets. It also confirms the idea of atoms as generators.

A Some general remarks

- Finite Complete Boolean Algebra are evidently atomic, thus their cardinality is a power of two (as they are isomorphic to the powerset of their atoms).
- $\emptyset$ is not a Boolean Algebra, as it has neither minimal object nor maximal object.
- $\{*\}$ where $* = 0 = 1$ is the most simple Boolean Algebra. It has Atoms $= \emptyset$.
- Next comes $\{0, 1\}$, which is in fact the most widely used Boolean Algebra. True and false are the only truth values in standard logic which plays a big role in computer science.
- Associated to Boolean Algebras are the Boolean rings with $x^2 = x$ for all x. The Boolean Ring to $\{0, 1\}$ is of course $\mathbb{F}_2$ which incidentally is the only the Boolean Ring that is an integral domain.
- As we have seen, the axioms of Boolean Algebra imply the rule of excluded middle.
- There are non-atomic, but complete Boolean Algebras. E.g. one can define an ordering on classes of logic formulas by using implication. But for any formula F, we consider a variable x which does not appear in F. Then $F \wedge x \Rightarrow F$, but not $F \Rightarrow F \wedge x$, so F and $F \wedge x$ have different classes, and $[F \wedge x] \leq [F]$.
- As Boolean Algebras are categories, they can be seen as graphs as demonstrated in the figure.

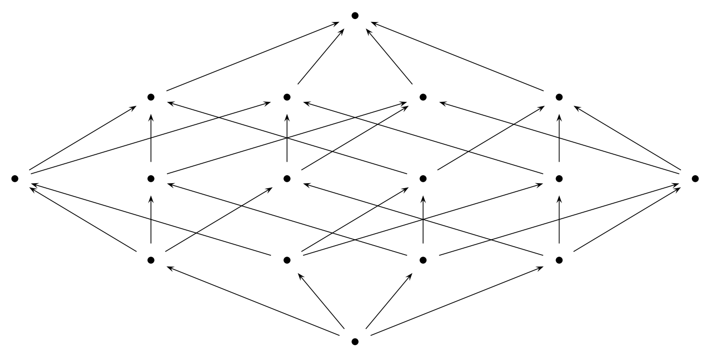

Figure 1: Graph of a Complete Boolean Algebra with four atoms

References

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[2] S. Mac Lane. *Categories for the working mathematician*, page 95ff. Springer Verlag, second edition, 1998.

[3] Jaap van Oosten. Basic category theory. http://www.math.uu.nl/people/jvoosten/, July 2002. Lecture notes on Category Theory.